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On the impulsive motion of a flat plate in a viscous fluid

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The unsteady laminar boundary-layer flow induced by the impulsive motion of a semi-infinite flat plate along its length is investigated. It is found that, unlike Stewartson's (1951, 1960) conclusion, a power-series solution is possible using the 'correctly stretched' variables in the analysis. The small-time solution, which is developed in powers of the time, shows a smooth transition from the initial Rayleigh flow to the final Blasius flow without an essential singularity and, furthermore, its validity extends to the whole time domain. The series solution developed for large times, however, seems divergent and merely asymptotic. No evidence is found for the existence of an essential singularity in the solution as described by Stewartson.

1. Introduction

This paper examines the unsteady laminar boundary-layer flow generated by a semi-infinite flat plate which starts to move impulsively from rest parallel to its length with constant velocity U. The problem is a basic one in unsteady viscous flow theory and also has practical importance in shock-tube applications (Lam & Crocco 1958). In spite of the extensive works on the subject in the past, the problem still remains intriguing and confusing owing to mathematical difficulties in the solution which are not present when the body is bluff.

Blasius (1908) and Goldstein & Rosenhead (1936), for example, considered the impulsive motion of bluff bodies such as cylinders for which the external inviscid velocity U outside the boundary layer is not constant. Initially, the boundary layer has zero thickness and therefore, at the beginning of the motion, the diffusion is far greater than the convection and the influence of the pressure gradient. In the first approximation, Goldstein & Rosenhead obtain the solution to a diffusion equation which results after neglecting the non-linear convection and pressure gradient terms in unsteady boundary-layer equations, and then iterate to obtain higher approximations. The Goldstein–Rosenhead solution is obtained in powers of the time and should be valid for small times extending to a nearest singularity in the solution. With only three terms of the expansion known so far, the radius of convergence cannot be determined with any certainty.

A similar approach fails, however, for the present case of a semi-infinite flat plate for which the velocity outside the boundary layer is constant. All terms other than the first in the infinite series of the Goldstein–Rosenhead solution become zero, the first term being the well-known Rayleigh solution for an infinite flat plate. After a long time, however, the flow must settle down to the steady Blasius flow. A question now remains how the initial Rayleigh solution having no x-dependence could possibly settle down to the x-dependent but time-independent Blasius solution. Stewartson (1951, 1960) was the first to attempt a plausible explanation for it. From the physical requirement, the Rayleigh flow must shift to an x-dependent Blasius-type flow in a smooth manner. Stewartson argued then that, in the light of no analytical development from Rayleigh- to Blasius-type flows, the smooth transition can only be possible through an essential singularity. Rather intuitively Stewartson considers $\tau = 1$ as the location of this singularity, where $\tau = Ut/x$, x being the distance measured from the leading edge and t the time elapsed since the start of the motion. It is argued that, for $\tau \ll 1$, since a disturbance at the leading edge has not arrived yet, the flow remains of Rayleigh type; however, at $\tau = 1$, the leading-edge effect is felt suddenly and the flow starts to have x-dependence by means of an essential singularity.† The attempt to construct the mathematical evidence for the existence of such a singularity was made by Stewartson for a flat plate and by Smith (1967) for a wedge generalizing Stewartson's analysis. As a first step, Stewartson and Smith show that solutions to strongly simplified approximate equations of motion, one by Oseen's approximation method and the other by momentum integral method, exhibit the anticipated singular behaviour. However, their attempts to construct essential singularities in the solution using complete equations of boundary-layer motion fail since some assumptions used are still to be justified. Stewartson therefore concludes the existence of singularities in this region as tentative only. Despite further investigations by Schuh (1953), Cheng (1957) and Cheng & Elliot (1957), no new features were uncovered to change Stewartson's conclusion substantially. Indeed, a uniqueness theorem given by Lam & Crocco (1958) for the problem in this region seemed sufficient to establish the validity of Stewartson's solution with the essential singularity at $\tau = 1$—see a recent article by Rott (1964, p. 425), for example. If so, up to the point of singularity $\tau = 1$, the first-term Rayleigh's solution is the exact solution of the problem.

One now turns to the flow behaviour when the time elapsed after the start of impulsive motion becomes large. At sufficiently large times, the flow settles down to a steady flow everywhere or to the Blasius flow for a flat plate. Here again, no power-series solution can be found in inverse powers of the time, indicating another possible essential singularity (Stewartson 1951). Ironically, the similar situation arises even with bluff bodies this time (Kelly 1962; Smith 1967). The analyses of Stewartson for a flat plate and those of Kelly and Smith for bluff bodies at large times show that the approach to a steady state is exponential. This seems to indicate that an analytic power-series solution may not exist to the problem of impulsive motion regardless of body shapes at large times. Interestingly enough, however, power-series solutions in inverse of the time exist at large times if the body moves continuously, for example, with constant acceleration. Cheng & Elliot (1957) obtained such series solution for a flat plate and Tokuda & Yang (1966) for stagnation flows. Furthermore, the result

† This argument is questionable. See the discussion in § 6.1.

of Tokuda & Yang for stagnation flow with constant acceleration shows that the small- and large-time series solutions join smoothly in the intermediate time, indicating no occurrence of a singularity in the solution for the whole time domain. For the impulsive motion of a flat plate, the first term of the Blasius solution is no longer an exact solution at large times as the Rayleigh solution is at small times. Stewartson's solution tends to the Blasius flow with exponentially small terms. Assuming that Stewartson's conclusion on small- and large-time solutions is correct, Lam & Crocco (1958) and Akamatsu & Kamimoto (1966) investigated the behaviour of the solution between $\tau = 1$ and $\tau = \infty$. In this region, then, the solution changes from the initial Rayleigh flow to an x-dependent Blasius-type flow by means of an essential singularity at $\tau = 1$ and approaches the Blasius flow exponentially toward an infinite time. Lam & Crocco transform the governing equation into an integral equation and seek the solution as the converged limit of the iterants. The iterants seem to converge at around the 14th iteration but are reported to diverge after the 18th iteration especially near the region of singularity $\tau = 1$. Akamatsu & Kamimoto (1966) employ Meksyn's (1961) method to seek the asymptotic form of the solution. Here again one notices an unnecessarily large deviation of the result from the Blasius solution at $\tau = \infty$ if the condition at the singularity $\tau = 1$ is imposed and the modification will be discussed further in §6.2. Their solution shows again the exponential approach to the Blasius flow.

As described in detail in the preceding paragraphs, some doubt remains about Stewartson's conclusion on the essential singularity in the solution. Indeed, the present analysis shows that the solution has a power-series development for both small and large times without encountering an essential singularity if one chooses the 'appropriately stretched' co-ordinate system. This co-ordinate system will be deduced from physical arguments after examining the vorticity transport mechanism within the fluid.

2. Physical background of unsteady viscous flows

In this section the general nature of unsteady viscous flow induced by the impulsive motion of a flat plate will be examined by simple physical reasoning. Vorticity considerations developed here are seen not only to illuminate the detailed development of the boundary layer but also to provide the basis for deducing the appropriate form of stretched variables to use in the analysis. The role of vorticity transfer is essential in the development of boundary-layer theory. Indeed, Lighthill (1963) shows convincingly that, to determine the whole viscous flow field completely, it is sufficient to study the development of the vorticity field. It is then anticipated that the nature of viscous flow solution strongly depends on the dominating mode of vorticity propagation away from its source (i.e. solid boundaries) through the otherwise undisturbed fluid. It is demonstrated that the qualitative nature of a flow can be established first by identifying, and then by examining the magnitude of, dimensionless parameters relevant to vorticity transfer in the problem. This approach was demonstrated systematically first by Stuart (1963) in the investigation of unsteady boundary layers.

Let a semi-infinite flat plate be immersed in a quiescent, incompressible fluid of vanishingly small viscosity and let its leading edge be the origin of a rectangular co-ordinate system (x, y) with the plate occupying the positive x-axis (see figure 1). At the instant $t = 0$, the plate moves impulsively from rest with uniform velocity U in the negative x-direction through otherwise undisturbed fluid. The plate wall then becomes a source of vorticity due to the no-slip condition. At $t = 0$, the vorticity generated along the plate wall is concentrated in the form of a vortex sheet which will be diffused out from the wall by viscous diffusion and convected

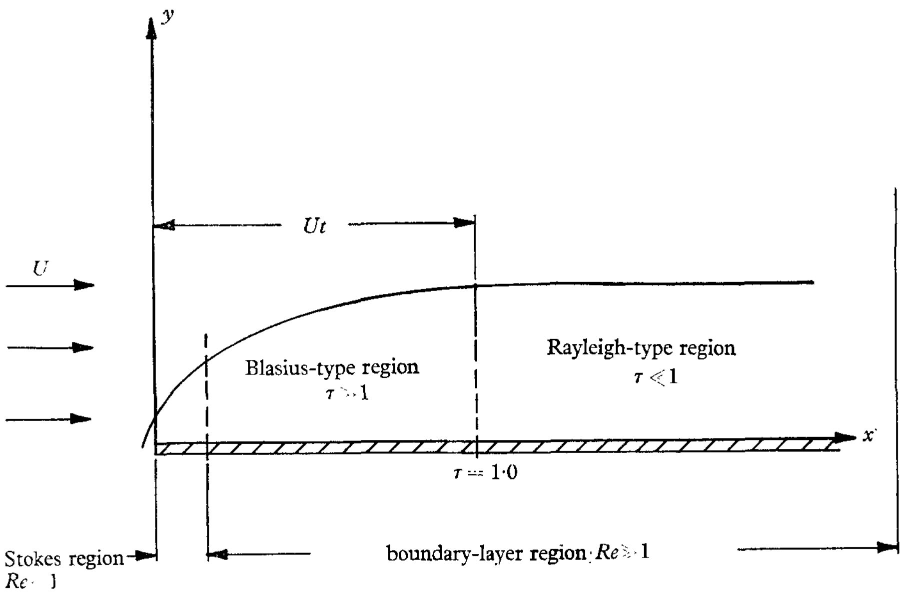

FIGURE 1. Region of unsteady laminar boundary layer past a flat plate.

simultaneously by associated flows through the fluid after a short time, forming a vorticity layer. Let δ be the thickness of the vorticity layer formed over the plate, which generally depends on both x and t and ν, the kinematic viscosity of the fluid. Then the rate with which the bulk properties of the fluid such as vorticity or momentum diffuse away from the wall in the transverse direction is of the order of ν/δ^2. This may be conveniently called the 'transverse diffusion rate' in analogy with the diffusion time, δ^2/ν, of Stuart (1963, p. 349). In the present problem, there exist three characteristic rate scales, namely $1/t$, ν/x^2 and U/x, which are associated with vorticity transfer through the viscous fluid and the most dominant of which is required to be of the same order of the transverse diffusion rate. This requirement is familiar in Prandtl's boundary-layer theory if the no-slip condition is to be satisfied on the wall. Indeed, depending on which one of the three dominates the others and balances the transverse diffusion rate, the familiar Rayleigh, Stokes and Blasius flows are obtained as follows:

$$\left.\begin{array}{rlll} \text{Rayleigh flow:} & Re/\tau \gg 1, \quad \tau \ll 1 & \text{and} & \delta = O([\nu t]^{\frac{1}{2}}), \\ \text{Stokes flow:} & Re/\tau \ll 1, \quad Re \ll 1 & \text{and} & \delta = O(x), \\ \text{Blasius flow:} & \tau \gg 1, \quad Re \gg 1 & \text{and} & \delta = O([U/\nu x]^{\frac{1}{2}}), \end{array}\right\} \quad (2.1)$$

where $Re = Ux/\nu$ and $\tau = Ut/x$. Equation (2.1) shows how the nature of flow changes according to the magnitude of Re and τ. For example, for a small Re fixed, the flow changes from Rayleigh- to Stokes-type flow according as $\tau \ll 1$ or $\tau \gg 1$. In a similar manner for a large Re fixed, which is the subject of this paper, the change is from Rayleigh to Blasius flows. See the schematic diagram in figure 1 for details. This argument can be made more rigorous if we write

$$\nu/\delta^2 = 1/t + \nu/x^2 + U/x. \tag{2.2}$$

Physically (2.2) can be interpreted as expressing the conservation equation of vorticity in a qualitative manner. ν/δ^2 represents the order of rate of vorticity generation at the wall, while $1/t$ and ν/x^2 represent the rate of vorticity diffusion, the former according to a Gaussian distribution (Lighthill 1963, p. 56) and the latter in the longitudinal direction, and U/x represents the rate of vorticity transport by convection. Equation (2.2) may be rewritten in the following forms according to the magnitude of Re as

$$\delta/x = \frac{1}{Re^{\frac{1}{2}}}\frac{1}{(1+1/\tau+1/Re)^{\frac{1}{2}}} \quad \text{for} \quad Re \gg 1, \tag{2.3a}$$

$$= \frac{1}{(1+Re/\tau+Re)^{\frac{1}{2}}} \quad \text{for} \quad Re \ll 1. \tag{2.3b}$$

In the present paper, consideration will be given only to the case $Re \to \infty$ or the boundary-layer region proper. Therefore, one may be justified in neglecting the $1/Re$ term in comparison with others of (2.3*a*). Now, depending on the magnitude of τ, δ may be rearranged from (2.3*a*) as

$$\delta = \frac{(\nu t)^{\frac{1}{2}}}{(1+\tau)^{\frac{1}{2}}} \quad \text{for} \quad \tau \ll 1, \tag{2.4a}$$

$$= \frac{x}{Re^{\frac{1}{2}}}\frac{1}{(1+1/\tau)^{\frac{1}{2}}} \quad \text{for} \quad \tau \gg 1. \tag{2.4b}$$

The form of δ in (2.4) suggests that the original Rayleigh and Blasius boundary-layer thickness is only the first term of the asymptotic expansion as $\tau \to 0$ and $\tau \to \infty$ respectively with $Re \to \infty$.

In singular perturbation problems such as the boundary-layer flows, the thin singular region must be accordingly stretched. The choice of stretching is often critical for the solution. It is most plausible then to use δ in (2.4) in forming the stretched variables. The stretched variable $Y = y/\delta$ to use in the later analysis may most conveniently be given as

$$Y = \frac{y}{(\nu t)^{\frac{1}{2}}}(1+\tau)^{\frac{1}{2}} \quad \text{for Rayleigh-type flow as} \quad \tau \to 0, \tag{2.5a}$$

and

$$Y = y\left(\frac{U}{\nu x}\right)^{\frac{1}{2}}(1+1/\tau)^{\frac{1}{2}} \quad \text{for Blasius-type flow as} \quad \tau \to \infty. \tag{2.5b}$$

All the previous analyses, including those of Stewartson (1951) and Lam &

Crocco (1958), among others, employ the original Rayleigh and Blasius stretched variables, $y/(\nu t)^{\frac{1}{2}}$ and $y(U/\nu x)^{\frac{1}{2}}$, respectively, and attempt to expand the solution in powers of τ or $1/\tau$. Writing $\eta = y/(\nu t)^{\frac{1}{2}}$, the present variable Y is given as $\eta(1+\tau)^{\frac{1}{2}}$ for small τ. This shows that a perturbation of the separable type in Stewartson's analysis becomes invalid as $\eta \to \infty$ even for $\tau \to 0$. τ included in the stretched variable Y becomes a forcing function for bringing x- or t-dependence into the Rayleigh or Blasius flow respectively and an analytic solution changing smoothly from the Rayleigh to Blasius flow becomes possible.

3. Governing equations and boundary conditions

This paper is concerned with the flow where $Re \gg 1$. Therefore, the appropriate governing equations are those of Prandtl's boundary-layer flows,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{3.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.2}$$

The rectangular co-ordinates (x, y) are those introduced in §2 and (u, v) represent the velocity components in these directions.

The boundary conditions to be used with these equations are

$$u = v = 0 \qquad \text{for} \quad x > 0,† \quad y = 0, \quad t > 0, \tag{3.3}$$

$$u \to U \quad \text{exponentially for} \quad x > 0, \quad y > 0, \quad t \to 0, \tag{3.4}$$

$$u \to U \quad \text{exponentially for} \quad x > 0, \quad y \to \infty, \quad t > 0. \tag{3.5}$$

The exponential approach in (3.4) and (3.5) is exhibited in the Oseen-type solutions of Stewartson (1951) and Smith (1967) and must be imposed as an additional condition for flows past bodies with a sharp leading edge (see Van Dyke 1964*a*, p. 139; 1964*b*).

To solve equations (3.1) to (3.5), we introduce the stream function ψ and stretched variable Y and τ,

$$\psi = \left[\frac{2\nu x}{U(1+\tau)}\right]^{\frac{1}{2}} Uf(Y, \tau), \tag{3.6}$$

$$Y = \frac{y}{(2\nu t)^{\frac{1}{2}}}(1+\tau)^{\frac{1}{2}} = y\left(\frac{U}{2\nu x}\right)^{\frac{1}{2}}(1+1/\tau)^{\frac{1}{2}}, \tag{3.7}$$

$$\tau = Ut/x. \tag{3.8}$$

† The analysis here is valid for $Re \gg 1$. Therefore, boundary conditions on x are to be imposed only for $x \gg \nu/U$. The flow becomes singular as $x \to 0$ so that $Re = O(1)$ where Stokes flow analysis must be employed (see Tokuda 1968). Within the framework of boundary-layer approximations, the boundary condition on $x = 0$ can certainly not be enforceable (cf. Stewartson 1960).

In terms of new variables, (3.1) and (3.2) now become

$$-\frac{Y}{1+\tau}f_{YY}+\frac{\tau}{(1+\tau)^2}Yf_{YY}+\frac{2\tau}{1+\tau}f_{Y\tau}-\frac{2\tau^2}{1+\tau}(f_Yf_{Y\tau}-f_{YY}f_\tau)$$
$$+\frac{\tau^2}{(1+\tau)^2}ff_{YY}=f_{YYY}. \quad (3.9)$$

where $$f_{Y\tau}=\frac{\partial^2 f}{\partial Y\partial\tau},\quad f_\tau=\frac{\partial f}{\partial\tau},\quad \dots$$

The boundary conditions for (3.9) are from (3.3) to (3.5):

$$f=f_Y=0,\quad Y=0,\quad \tau>0, \quad (3.10)$$

$$f_Y\to 1+\exp\quad \text{as}\quad Y\to\infty,\quad \tau>0. \quad (3.11)$$

Here 'exp' stands for terms that are exponentially small for large Y. Solutions for (3.9) satisfying boundary conditions (3.10) and (3.11) will be sought for small and large values of τ.

4. Solution for small τ

The flow approximates to the Rayleigh flow in the limit $\tau\to 0$. An asymptotic solution for small τ to the full boundary-layer equations (3.9) subject to (3.10) and (3.11) will be developed by a series expansion method.

An examination of (3.9) indicates, however, that, if the solution is developed in powers of τ, a singularity arises at $\tau=-1$, which inevitably restricts the convergence radius to $\tau=1$. However, this singularity can be eliminated. In the present problem, physical significance is attached only to the domain of positive real values of τ and no apparent physical reason can be given for the singularity at $\tau=-1$. Bellman (1955) and Van Dyke (1964*a, b*) showed that such an artificial singularity might be caused by an inappropriate choice of a co-ordinate system and could be eliminated by a simple transformation such as Euler's using the principle of analytic continuation. Therefore τ is now recast into ξ, using the Euler transformation, as

$$\xi=\frac{\tau}{1+\tau}. \quad (4.1)$$

In terms of Y and ξ, (3.9) now reduces to

$$-(1-\xi)^2Yf_{YY}+2\xi(1-\xi)^2f_{Y\xi}-2\xi^2(1-\xi)(f_Yf_{Y\xi}-f_{YY}f_\xi)-\xi^2ff_{YY}=f_{YYY}. \quad (4.2)$$

Equation (4.2) clearly shows that the artificial singularity is eliminated. Appropriate boundary conditions are

$$f=f_Y=0,\quad Y=0,\ 0<\xi<1, \quad (4.3)$$

$$f_Y\to 1+\exp,\quad \text{as}\quad Y\to\infty,\ 0<\xi<1. \quad (4.4)$$

One now assumes that the function $f(Y,\xi)$ may be expanded into a power series of ξ as

$$f(Y,\xi) = f_0(Y)+\xi f_1(Y)+\xi^2 f_2(Y)+\sum_{n=3}^{\infty}\xi^n f_n(Y). \tag{4.5}$$

Substituting (4.5) into (4.2), (4.3) and (4.4) and collecting the like powers of ξ, one obtains the following:

$$\left.\begin{aligned}
&\xi^0, \\
&\qquad Yf_0''+f_0''' = 0, \\
&\xi^1, \\
&\qquad Yf_1''-2f_1'+f_1''' = 2Yf_0'', \\
&\xi^2, \\
&\qquad Yf_2''-4f_2'+f_2''' = Y(2f_1''-f_0'')-4f_1'-2(f_0'f_1'-f_0''f_1)-f_0f_0''.
\end{aligned}\right\} \tag{4.6}$$

Generally

$$\xi^n, \qquad l_n(f_n) = g_n,$$

where

$$l_n = Y\frac{d^2}{dY^2}-2n\frac{d}{dY}+\frac{d^3}{dY^3},$$

$$\begin{aligned}
g_n = {} & Y(2f_{n-1}''-f_{n-2}'')-4(n-1)f_{n-1}' \\
& +2(n-2)f_{n-2}'-2\sum_{r=0}^{n-2}(n-1-r)\{f_r'f_{n-1-r}'-f_r''f_{n-1-r}\} \\
& +2\sum_{r=0}^{n-2}(n-2-r)\{f_r'f_{n-2-r}'-f_{r-1}''f_{n-2-r}\}-\sum_{r=0}^{n-2}f_rf_{n-2-r}''.
\end{aligned}$$

Boundary conditions are

$$\left.\begin{aligned}
& f_0(0) = f_0'(0) = 0, \quad f_0'(\infty) = 1+\text{exp} \\
\text{and} \quad & f_n(0) = f_n'(0) = 0, \quad f_n'(\infty) = 0+\text{exp} \quad \text{for} \quad n = 1, 2, 3, \ldots.
\end{aligned}\right\} \tag{4.7}$$

f_0 can be readily integrated to give $f_0' = \text{erf}\,(Y/\sqrt{2})$. Hence $f_0' \sim 1+\text{exp}$ as $Y \to \infty$. f_n functions in (4.6), satisfying $l_n(f_n) = g_n$, are known as parabolic cylindrical functions. For $n = 1$, $g_1 \to 0+\text{exp}$ as $Y \to \infty$. The asymptotic solution of f_1 as $Y \to \infty$ is well known and can be given as (see Jones & Watson 1963, p. 247) $f_1' \sim A\exp(-Y^2/2)\,Y^{-3}+BY^2$ as $Y \to \infty$, where A and B are arbitrary constants. We must choose $B = 0$. Hence $f_1' \sim 0+\text{exp}$ as $Y \to \infty$. It is easy to show, by mathematical induction, that $f_n' \to 0+\text{exp}$ for any $n = 1, 2, 3, \ldots$. Therefore, the exponential approach for f_n functions is assured. Differential equations for f_n are all linear and may in principle be obtained as combinations of the error function and its integrals. In the present paper, however, because of tediousness in calculations for higher orders, the f_n functions are integrated numerically by the Runge–Kutta method on the Univac 1108 computer to $n = 8$. Double precision is employed throughout the calculation. Step size of integration is 0·01 and the integration was carried out up to $Y = 7{\cdot}0$. The result of the calculation will be discussed, together with the large-time solution, in §6. One now turns to the solution for large τ.

5. Solution for large τ

For $\tau \gg 1$, the flow should settle down to the Blasius flow everywhere. For this time domain, it is most appropriate to apply the Euler transformation in terms of $1/\tau$ as

$$\bar{\xi} = \frac{1/\tau}{1+1/\tau} = 1-\xi. \tag{5.1}$$

Hence, as $\tau \to \infty$, $\xi \to 1$ and $\bar{\xi} \to 0$. An asymptotic solution will therefore be sought in the limit as $\bar{\xi} \to 0$.

In terms of Y and ξ, (3.9) now reduces to

$$-\bar{\xi}^2 Y f_{YY} - 2\bar{\xi}^2(1-\bar{\xi}) f_{Y\bar{\xi}} + 2\bar{\xi}(1-\bar{\xi})^2 (f_Y f_{Y\bar{\xi}} - f_{YY} f_{\bar{\xi}}) - (1-\bar{\xi})^2 f f_{YY} = f_{YYY}. \tag{5.2}$$

Boundary conditions are

$$f = f_Y = 0, \quad Y = 0,\ 0 < \bar{\xi} < 1, \tag{5.3}$$

$$f_Y \to 1 + \exp, \quad \text{as} \quad Y \to \infty,\ 0 < \bar{\xi} < 1. \tag{5.4}$$

Since Y does not change whether $\tau \ll 1$ or $\tau \gg 1$, (5.2) is deduced from (4.2) by substituting $\xi = 1-\bar{\xi}$. One again assumes that the function $f(Y, \bar{\xi})$ may be expanded into a power series of $\bar{\xi}$ as

$$f(Y, \bar{\xi}) = F_0(Y) + \bar{\xi} F_1(Y) + \bar{\xi}^2 F_2(Y) + \sum_{n=3}^{\infty} \bar{\xi}^n F_n(Y). \tag{5.5}$$

F_n functions must obey the following equations:

$$\left.\begin{aligned}
&\bar{\xi}^0, \\
&\qquad F_0 F_0'' + F_0''' = 0, \\
&\bar{\xi}^1, \\
&\qquad F_1 F_0'' + F_0 F_1'' + F_1''' - 2(F_0' F_1' - F_0'' F_1) = 2F_0 F_0'', \\
&\bar{\xi}^2, \\
&\qquad F_0 F_2'' + F_2 F_0'' + F_2''' - 4(F_0' F_2' - F_0'' F_2) = -Y F_0'' - 2F_1' \\
&\qquad\quad - 4(F_0' F_1' - F_0'' F_1) + 2(F_1' F_1' - F_1'' F_1) + 2(F_0 F_1'' + F_1 F_0'') \\
&\qquad\qquad - F_0 F_0'' - F_1 F_1''.
\end{aligned}\right\} \tag{5.6}$$

Generally

$$\bar{\xi}^n, \qquad L_n(F_n) = G_n,$$

where

$$L_n = F_0'' - 2n\left(F_0' \frac{d}{dY} - F_0''\right) + F_0 \frac{d^2}{dY^2} + \frac{d^3}{dY^3},$$

$$\begin{aligned}
G_n = {} & -Y F_{n-2}'' - 2\{(n-1)F_{n-1}' - (n-2)F_{n-2}'\} - 4\sum_{r=0}^{n-2} (n-r-1)(F_r' F_{n-r-1} \\
& - F_r'' F_{n-r-1}) + 2\sum_{r=0}^{n-3} (n-r-2)(F_r' F_{n-r-2}' - F_r'' F_{n-r-2}) \\
& + 2\sum_{r=0}^{n-4} (n-r-3)(F_r' F_{n-r-3}' - F_r'' F_{n-r-3}) - \sum_{r=0}^{n-3} F_r F_{n-r-2}'' \\
& + 2\sum_{r=0}^{n-1} F_r F_{n-r-1}'' - \sum_{r=0}^{n-2} F_r F_{n-r-2}''.
\end{aligned}$$

Boundary conditions are

$$F_0(0) = F_0'(0) = 0, \quad F_0'(\infty) = 1+\exp,$$

and $$F_n(0) = F_0'(0) = 0, \quad F_0'(\infty) = 0+\exp, \quad \text{for} \quad n = 1, 2, 3, \ldots \qquad (5.7)$$

F_0 is the well-known Blasius function as expected. For large values of Y,

$$F_0' \sim 1+\exp \quad \text{and} \quad F_0 \sim Y-\beta+\exp,$$

where $\beta = 1{\cdot}21678$. Therefore, for large Y, F_n functions in (5.6) behave like parabolic cylindrical functions and their exponential approach as $Y \to \infty$ is assured as for f_n functions. The F_n functions are integrated numerically using the identical step size and integration domain as in f_n functions.

6. Results and discussion

Complete numerical results obtained using double-precision calculation for f_n and F_n functions ($n = 0, 1, \ldots, 8$) will not be given in this paper in the interests of brevity. They may be obtained from the author upon request. Table 1 summarizes some important results.

n	$f_n''(0)$	$f_n(\infty)$†	$F_n''(0)$	$F_n(\infty)$†
0	0·797885	6·102115	0·469600	5·683219
1	−0·400640	−0·401719	−0·234806	−0·610899
2	−0·622220 × 10⁻¹	−0·242945	−0·573581 × 10⁻¹	−0·460303
3	0·801778 × 10⁻²	−0·111324	−0·294199 × 10⁻¹	−0·388054
4	0·422265 × 10⁻¹	0·898340 × 10⁻¹	−0·209109 × 10⁻¹	−0·349769
5	0·623748 × 10⁻¹	0·117461	−0·245675 × 10⁻¹	−0·342269
6	0·742694 × 10⁻¹	0·211513	−0·573250 × 10⁻¹	−0·399750
7	0·802521 × 10⁻¹	0·289651	−0·202173	−0·675072
8	0·817048 × 10⁻¹	0·352695	−0·825587	−1·808625

† $Y = 6{\cdot}9$ is taken as a point at ∞.

TABLE 1. Summary for f_n and F_n functions

The skin friction coefficient may be found then as

$$S = C_f(Re/2)^{\frac{1}{2}} \sim \frac{1}{\xi^{\frac{1}{2}}}[0{\cdot}797885 - 0{\cdot}400640\xi - 0{\cdot}062222\xi^2 + 0{\cdot}008018\xi^3 + 0{\cdot}042226\xi^4$$
$$+ 0{\cdot}062375\xi^5 + 0{\cdot}074269\xi^6 + 0{\cdot}080252\xi^7 + 0{\cdot}081705\xi^8 + O(\xi^9)] \qquad (6.1)$$

for a Rayleigh-type flow in which ξ is small and

$$S = \frac{1}{(1-\bar{\xi})^{\frac{1}{2}}}[0{\cdot}469600 - 0{\cdot}234806\bar{\xi} - 0{\cdot}057358\bar{\xi}^2 - 0{\cdot}029420\bar{\xi}^3 - 0{\cdot}020911\bar{\xi}^4$$
$$- 0{\cdot}024568\bar{\xi}^5 - 0{\cdot}057325\bar{\xi}^6 - 0{\cdot}202173\bar{\xi}^7 - 0{\cdot}825587\bar{\xi}^8 + O(\bar{\xi}^9)] \qquad (6.2)$$

for a Blasius-type flow in which $\bar{\xi}$ is now small. Here $C_f = 2\tau_w/\rho U^2$, $Re = Ux/\nu$, $\xi = \tau/(1+\tau)$ and $\bar{\xi} = 1-\xi$.

The series solutions of (6.1) and (6.2) are examined carefully in the following. Contrary to Stewartson's conclusion, an analytic solution with smooth transition from Rayleigh to Blasius flow is obtained.

6.1. *Rayleigh-type solution near* $\xi \ll 1$

Using the expansion $(1-\xi)^{-\frac{1}{2}} = 1+\frac{1}{2}\xi+\frac{3}{8}\xi^2+\frac{5}{16}\xi^3+\ldots$, it is convenient to rewrite (6.1) as

$$S = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(1-\xi)^{\frac{1}{2}}}{\xi^{\frac{1}{2}}}[1-0{\cdot}002127\xi+0{\cdot}046952\xi^2+0{\cdot}095360\xi^3+0{\cdot}145227\xi^4$$
$$+0{\cdot}177029\xi^5+0{\cdot}227952\xi^6+0{\cdot}266916\xi^7+0{\cdot}287808\xi^8+O(\xi^9)]. \quad (6.3)$$

Note here that $(2/\pi)^{\frac{1}{2}} = 0{\cdot}797885$ and the first term of (6.3), $([2/\pi][\xi/\{1-\xi\}])^{\frac{1}{2}}$, corresponds to the exact Rayleigh solution, which, according to Stewartson (1951) and others, must be the exact solution for $0 \leqslant \xi \leqslant \frac{1}{2}$. It is significant to note, however, that the series (6.3) has a non-trivial analytical expansion at $\xi = 0$ from the Rayleigh solution and furthermore the convergence of the series seems to extend to $\xi = 1$ or the whole time domain of interest. The series does converge rather slowly with nine terms available beyond $\xi = 0{\cdot}6$. To obtain reliable values, a method for improving the series will be applied here. Shanks (1955) sets forth a remarkable scheme of non-linear transformation to accelerate the convergence of some slowly convergent or even divergent series. The transformation is given as

$$\rho_k(S_n) = \frac{\rho_{k-1}(S_{n+1})\rho_{k-1}(S_{n-1})-\rho_{k-1}(S_n)^2}{\rho_{k-1}(S_{n+1})+\rho_{k-1}(S_{n-1})-2\rho_{k-1}(S_n)} \quad \text{for} \quad k = 1, 2, 3, \ldots$$

and
$$\rho_0(S_n) = S_n, \quad (6.4)$$

where S_n is the sum of the first n terms of the series and k is the number of iterations for which the Shanks transformation is applied. Shanks's method has been applied by the author quite successfully in other contexts and will be discussed in detail elsewhere. In table 2 the variation of skin friction from the

ξ	S_9	$\rho_1(S_9)$	$\rho_3(S_9)$	$\rho_3(S_9)$	Stewartson's (1951) analysis
0	0·797885	0·797885	0·797885	0·797885	0·797885
0·1	0·798171	0·798171	0·798171	0·798171	0·797885
0·2	0·799870	0·799870	0·799870	0·799870	0·797885
0·3	0·804245	0·804248	0·804248	0·804248	0·797885
0·4	0·813837	0·813884	0·813880	0·813880	0·797885
0·5	0·833971	0·843439	0·834397	0·834390	0·797885
0·6	0·876330	0·879735	0·879376	0·879318	—
0·7	0·968331	0·989661	0·987168	0·987524	—
0·8	1·183770	1·318334	1·307624	1·393594	—
0·9	1·803497	3·02094	3·97099	3·204714	—

TABLE 2. Skin friction, $S\xi^{\frac{1}{2}}/(1-\xi)^{\frac{1}{2}}$, from Rayleigh series of (6.3)

Rayleigh value, $S\xi^{\frac{1}{2}}/(1-\xi)^{\frac{1}{2}}$, is given for several values of ξ using all 9 terms of (6.3) along with the improved values by Shanks's transformation. Converged values are obtained up to $\xi = 0{\cdot}7$. However, beyond $\xi = 0{\cdot}8$, even with the Shanks transformation, no reliable values are obtained. A similar scatter is observed even for a well-defined convergent series if only finite terms of the series are used. We choose the following simple function $h(\xi)$ for demonstration,

$$h(\xi) = \exp(\xi/(\xi-1)) \quad (0 \leqslant \xi < 1). \tag{6.5}$$

$h(\xi)$ is analytic everywhere for $0 \leqslant \xi < 1$ and may therefore be represented by the following convergent power series:

$$h(\xi) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\left(\frac{\xi}{1-\xi}\right)^n \tag{6.6a}$$

$$= 1 - \xi - 0{\cdot}5\xi^2 - 0{\cdot}166667\xi^3 + 0{\cdot}041667\xi^4 + 0{\cdot}15\xi^5 + 0{\cdot}209722\xi^6 + 0{\cdot}216468\xi^7 + 0{\cdot}194969\xi^8 + O(\xi^9). \tag{6.6b}$$

Shanks's transformation is applied to the nine terms of the series of (6.6). The result is compared with the exact value of $h(\xi)$ in table 3. Using (6.6*b*), the con-

ξ	Exact $h(\xi)$	S_9	$\rho_1(S_9)$	$\rho_2(S_9)$	$\rho_3(S_9)$
0·0	1·00	1·0	1·0	1·0	1·0
		(1·0)	(1·0)	(1·0)	(1·0)
0·1	0·894840	0·894840	0·894840	0·894840	0·894840
		(0·894840)	(0·894840)	(0·894840)	(0·894840)
0·2	0·778801	0·778800	0·778801	0·778801	0·778801
		(0·778798)	(0·778798)	(0·778798)	(0·778798)
0·3	0·651439	0·651440	0·651440	0·651440	0·651440
		(0·651415)	(0·651419)	(0·651419)	(0·651419)
0·4	0·513417	0·513416	0·513416	0·513416	0·513416
		(0·513277)	(0·513349)	(0·513343)	(0·513345)
0·5	0·367879	0·367882	0·367879	0·367879	0·367879
		(0·367186)	(0·367810)	(0·367757)	(0·367845)
0·6	0·223130	0·223222	0·223122	0·223131	0·223131
		(0·220175)	(0·223994)	(0·223756)	(0·226633)
0·7	0·0969720	0·101537	0·0966166	0·0969996	0·0969703
		(0·0867595)	(0·105755)	(0·105505)	(0·343082)
0·8	0·0183156	0·5301587	−0·011640	−0·020137	0·0182058
		(−0·006114)	(0·0772436)	(0·088424)	(−0·2301184)
0·9	0·000123410	54·9010	−16·17656	0·512823	−0·01734
		(−0·011885)	(0·341625)	(0·629603)	(−0·404993)

Values without parentheses are calculated from (6.6*a*) and with parentheses from (6.6*b*).

TABLE 3. $h(\xi) = e^{-\xi/(1-\xi)}$ from power series of (6.6*a*) and (6.6*b*)

vergence deteriorates beyond $\xi = 0{\cdot}7$ as indicated by divergent values of higher Shanks's transformation. Curiously enough, the Shanks method is extremely effective in extracting accurate values from the series of (6.6*a*) despite the fact that the original series (6.6*a*) is more divergent than (6.6*b*), for example, at $\xi = 0{\cdot}8$.

It is now concluded, then, that the skin friction in table 2 is accurate at least to $\xi = 0{\cdot}7$. No evidence therefore exists for the essential singularity at $\xi = \frac{1}{2}$ or $\tau = 1$.

This analytical transition is physically more plausible in view of the dominant diffusion mechanism involved in the problem. At the start of the motion, a vortex sheet of constant strength will be created along the wall of the plate. For an infinite flat plate, this vortex sheet extends to infinity, with the diffusion of vorticity taking place only in the transverse direction. On the other hand, for a semi-infinite plate the vortex sheet terminates at the leading edge. The uniformity of vorticity distribution in the x-direction is now lost near the leading edge and the diffusion of vorticity will become x-dependent. A disturbance at the leading edge is induced essentially by this diffusion and therefore is considered to extend beyond the leading edge including the downstream region with smooth decay of its strength. This disturbance will perhaps be transported simultaneously by convection in the manner described by Stewartson (1960). From these considerations, Stewartson's explanation for a sudden transition of the flow type cited in § 1 is physically questionable. A similar argument has been given by Riley (1963) for unsteady heat transfer over a flat plate. Stewartson's analysis predicts a 4·5 % lower skin friction value at $\xi = \frac{1}{2}$, compared with the present result.

6.2. *Blasius-type solution near* $\bar{\xi}(=1-\xi) \ll 1$

Equation (6.2) may be rewritten, using the $(1-\bar{\xi})^{-\frac{1}{2}}$ expansion, as

$$S \sim 0{\cdot}469600[1 - 0{\cdot}000013\bar{\xi} + 0{\cdot}0028512\bar{\xi}^2 + 0{\cdot}001227\bar{\xi}^3 - 0{\cdot}005723\bar{\xi}^4 - 0{\cdot}026871\bar{\xi}^5 - 0{\cdot}115362\bar{\xi}^6 - 0{\cdot}475604\bar{\xi}^7 - 2{\cdot}000963\bar{\xi}^8 + O(\bar{\xi}^9)]. \quad (6.7)$$

The first term of (6.7), 0·469600, corresponds to the exact Blasius value. Again it is seen that the skin friction series (6.7) has a power-series expansion in $\bar{\xi}$ around the Blasius solution. However, the series now seems divergent and merely asymptotic. Therefore, the large-time series should only be used for $\bar{\xi} \ll 1$. In table 4, the variation of skin friction for $\bar{\xi} = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$ is given. For $\bar{\xi} = 0{\cdot}1$ and $0{\cdot}2$, a point has not been reached yet beyond which the error starts to increase.

n	$\bar{\xi} = 0{\cdot}0$	$\bar{\xi} = 0{\cdot}1$	$\bar{\xi} = 0{\cdot}2$	$\bar{\xi} = 0{\cdot}3$
0	0·469600	0·495000	0·525029	0·561280
1	0·469600	0·470251	0·472525	0·477085
2	0·469600	0·469616	0·469959	0·470915
3	0·469600	0·469613	0·469696	0·469965
4	0·469600	0·469613	0·469659	0·469763
5	0·469600	0·469613	0·469650	0·469692
6	0·469600	0·469613	0·469645	0·469642
7	0·469600	0·469613	0·469643	0·469589
8	0·469600	0·469613	0·469641	0·469524

TABLE 4. Skin friction, S, from Blasius series of (6.7)

For $\bar{\xi} = 0{\cdot}3$, one must stop at $n = 6$ since the error neglected in the series starts to increase beyond it.

The analysis of Akamatsu & Kamimoto (1966), using Meksyn's (1961) asymptotic approach, is of interest. The essence of Meksyn's method is to expand the stream function for small Y and to determine the variable coefficients using the asymptotic method of steepest descent. It is most appropriate to impose the inner boundary conditions if known rather than the outer free-stream condition as $Y \to \infty$ where the expansion loses its validity. Accordingly, specifying the boundary conditions that S takes on the Blasius value $(= 0{\cdot}4696)$ at $\bar{\xi} = 0$ and the present Rayleigh-series value $(= 0{\cdot}8344)$ together with the continuous derivative at $\bar{\xi} = \frac{1}{2}$, the first-order solution of Akamatsu & Kamimoto (1966) for skin friction may be modified as

$$S = \left[1{\cdot}656 - 0{\cdot}527 \exp\left\{\frac{0{\cdot}65(2\bar{\xi}-1)}{\bar{\xi}}\right\}\right]^{-\frac{3}{2}}. \tag{6.8}$$

The result is given in figure 2 and good agreement with the present result is noted. On the other hand, Lam & Crocco's (1958) result based on Stewartson's analysis shows a 7–8 % deviation from the present result at $\xi = 0{\cdot}6$. The divergence of the iterants experienced by Lam & Crocco may be eliminated if the present small-time solution is used instead.

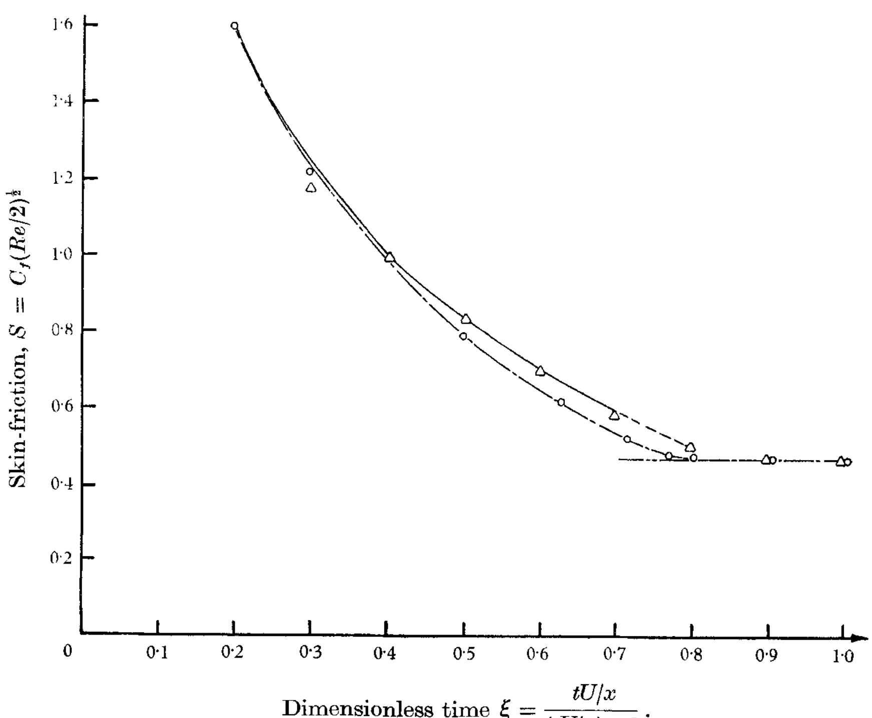

FIGURE 2. Variation of skin friction over a flat plate. ——, Rayleigh-series solution (6.1); — — — —, Blasius-series solution (6.2); ⊖ – ⊖, Lam & Crocco (1958); △ △, modified Akamatsu & Kamimoto equation (6.8).

7. Conclusions

The unsteady boundary-layer flow induced by the impulsive motion of a semi-infinite flat plate is re-examined. The use of the correctly stretched variable Y of (2.5) is the most important factor in the present analysis. The power-series solution for small ξ now becomes possible and the 9 terms of the series calculated seem to indicate its convergence in the whole domain of physical significance $0 \leqslant \xi < 1$. The appearance of a possible singularity at $\xi = \frac{1}{2}$ is eliminated by using a simple Euler transformation. No evidence for the type of an essential singularity at $\xi = \frac{1}{2}$ described by Stewartson (1951, 1960) is found. The power-series solution for large times or small $\bar{\xi}$ $(= 1 - \xi)$ seems merely asymptotic. The result of Akamatsu & Kamimoto (1966), using Meksyn's method, gives a good result if modified by imposing the inner boundary conditions rather than the outer condition. The Lam–Crocco result using Stewartson's solution up to $\xi = \frac{1}{2}$ deviates by 7 % from the present solution.

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